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ON NEAT EMBEDDINGS OF ALGEBRAISATIONS OF FIRST ORDER LOGIC

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Abstract

Let α be an infinite ordinal. There are non-isomorphic representable algebras of dimension α each of which is a generating subreduct of the same β dimensional algebra. Dually there exists a representable algebra $\mathfrak{A}$ of dimension α, such that $\mathfrak{A}$ is a generating subreduct of $\mathfrak{B}$ and $\mathfrak{B}'$, however, $\mathfrak{B}$ and $\mathfrak{B}'$ are not isomorphic. The above was proven to hold for infinite dimensional cylindric algebras (***CA's***) in [3] answering questions raised by Henkin et al. In this paper, we investigate the analogous statements for algebraisations other than cylindric algebras. We show that Pinter's substitution algebras and Halmos' quasi-polyadic algebras behave like ***CA****'s*, however, Halmos polyadic algebras do not.

1. Introduction

This paper is a follow up to [3]. We follow the notation adopted therein, which is in conformity with that adopted in the monograph [6]. The following (striking) result was proved in [3] confirming a conjecture of Tarski on cylindric algebras, cf. the introduction of [7]. **We cannot**

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replace $\boldsymbol{Dc}_\alpha$ in 2.6.67 (ii), 2.6.71-72 of [6] by $\boldsymbol{RCA}_\alpha$, when $\alpha \geq w$. In more detail, we have

Theorem 1. *For* $\alpha \geq w$, *the following hold*:

(i) *There are non-isomorphic representable cylindric algebras of dimension* α *each of which is a generating subreduct of the same* $\alpha + w$ *dimensional cylindric algebra.*

(ii) *There exist* $\mathfrak{A} \in \boldsymbol{RCA}_\alpha$, *a* $\mathfrak{B} \in \boldsymbol{CA}_{\alpha+w}$ *and an ideal* $J \subseteq \mathfrak{B}$, *such that* $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$, *A generates* $\mathfrak{B}$, *but* $\mathfrak{Ig}^{\mathfrak{B}}(J \cap A) \neq \mathfrak{B}$.

(iii) *There exist* $\mathfrak{A}, \mathfrak{A}' \in \boldsymbol{RCA}_\alpha$, $\mathfrak{B}, \mathfrak{B}' \in \boldsymbol{CA}_{\alpha+w}$ *with embeddings* $e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha\mathfrak{B}$ *and* $e_{A'} : \mathfrak{A}' \to \mathfrak{Nr}_\alpha\mathfrak{B}'$ *such that* $\mathfrak{Sg}^{\mathfrak{B}}e_A(A) = \mathfrak{B}$ *and* $\mathfrak{Sg}^{\mathfrak{B}'}e_{A'}(A) = \mathfrak{B}'$, *and an isomorphism* $i : \mathfrak{A} \to \mathfrak{A}'$ *for which there exists no isomorphism* $\bar{i} : \mathfrak{B} \to \mathfrak{B}'$ *such that* $\bar{i} \circ e_A = e_{A'} \circ i$.

The proof of Theorem 1, carried out in [3], depends on a deep result of Pigozzi, namely that $\boldsymbol{RCA}_\alpha$ does not have the amalgamation property (*AP*), when $\alpha \geq w$. [3] tells us where to find direct counterexamples, namely from common subalgebras of algebras in $\boldsymbol{RCA}_\alpha$, that do not amalgamate. In this paper, we show that an analogous result hold for Pinter's substitution algebras ***SC**'s* and Halmos' quasi-polyadic algebras with and without equality (***QEA***, ***QA***). Such algebras, together with their concrete versions, the so-called representable algebras, are defined in, e.g., [2] and [4]. For $\boldsymbol{K} \in \{\boldsymbol{SC}, \boldsymbol{CA}, \boldsymbol{QA}, \boldsymbol{QEA}\}$. $\boldsymbol{RK}_\alpha$ stands for the class of representable algebras in $\boldsymbol{K}_\alpha$, that is, those algebras that are isomorphic to subdirect products of set algebras. For a thorough treatment of such algebras, we refer to [4]. Neat reducts and neat embeddings for such algebras are defined like the ***CA*** case [1]. We give a contrasting result for polyadic algebras ***PA**'s*. For those we show that generating subreducts are rather neat reducts. However, unlike ***CA**'s*, for ***SC***, ***QA*** and ***QEA***, it is not known whether the class of representable algebras have *AP* or not. So to prove our first main result, we need

Lemma 2. *Let* α *be an infinite ordinal. Let* $\boldsymbol{K} \in \{\boldsymbol{SC}, \boldsymbol{QA}, \boldsymbol{CA}, \boldsymbol{QEA}\}$. *Then* $\boldsymbol{RK}_\alpha$ *does not have AP with respect to* $\boldsymbol{K}_\alpha$.

Proof. We prove that there are two representable algebras having a common subalgebra that cannot be embedded in a third algebra even, if the latter is sought in the bigger class $\boldsymbol{K}_\alpha$. We modify Pigozzi's proof proving the cylindric case [9], Theorem 2.3.6. Pigozzi's proof essentially depends on the existence of diagonal elements, our proof does not. In what follows by $\mathfrak{Sg}^{\mathfrak{A}}X$ or $\mathfrak{A}^{(X)}$, we denote the subalgebra of $\mathfrak{A}$ generated by X, and we write $\mathfrak{A}^{(x)}$ for $\mathfrak{A}^{(\{x\})}$. Seeking a contradiction, assume that $\boldsymbol{RK}_\alpha$ has *AP* with respect to $V = \boldsymbol{K}_\alpha$. Let $\mathfrak{A} = \mathfrak{Fr}_5 V$, the free V algebra on 5 generators. Let *r, s* and *t* be defined as follows:

$$r = \mathsf{c}_0(x \cdot \mathsf{c}_1 y) \cdot \mathsf{c}_0(x \cdot -\mathsf{c}_1 y),$$

$$s = \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 z \cdot \mathsf{s}_1^0 \mathsf{c}_1 z \cdot -m) + \mathsf{c}_0(x \cdot -\mathsf{c}_1 z),$$

$$t = \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 w \cdot \mathsf{s}_1^0 \mathsf{c}_1 w \cdot -m) + \mathsf{c}_0(x \cdot -\mathsf{c}_1 w),$$

where x, y, z, w and m are the first five generators of $\mathfrak{A}$. Here r, s and t are defined like in Pigozzi's case except that d_{01} is replaced by m, the fifth generator of $\mathfrak{Fr}_5 V$. (Diagonals will be introduced later in the proof). Then, exactly as in the proof of Lemma 2.3.1 in [9], one can show that $r \leq s \cdot t$. Let $X_1 = \{x, y\}$ and $X_2 = \{x, z, w, m\}$. Then

$$\mathfrak{A}^{(X_1 \cap X_2)} = \mathfrak{Sg}^{\mathfrak{A}}\{x\}. \tag{1}$$

We have

$$r \in A^{(X_1)} \text{ and } s, t \in A^{(X_2)}. \tag{2}$$

Let $\{x', y', z', w'\}$ be the first four generators of $\mathfrak{D} = \mathfrak{Fr}_4 \boldsymbol{RQEA}_\alpha$. Let h be the homomorphism from $\mathfrak{A}$ to $\mathfrak{Rd}_K\mathfrak{D}$, such that $h(i) = i'$ for $i \in \{x, y, w, z\}$ and $h(m) = \mathsf{d}_{01}$. Here $\mathfrak{Rd}_K\mathfrak{D}$ is the algebra obtained from $\mathfrak{D}$

by discarding operations not in the similarity type of $\boldsymbol{K}_\alpha$. Let J be the kernel of h. Then

$$\mathfrak{A} / J \cong \mathfrak{Rd}_K \mathfrak{D}. \tag{3}$$

We work inside the algebra $\mathfrak{A}$. Since $r \leq s \cdot t$, we have

$$r \in \mathfrak{Ig}^{\mathfrak{A}}\{s \cdot t\} \cap A^{(X_1)}. \tag{4}$$

Here, and elsewhere throughout the paper, $\mathfrak{Ig}^{\mathfrak{B}} X$ denotes the ideal generated by X. We shall use extensively that ideals function like the ***CA*** case. In particular, for $\mathfrak{A} \in \boldsymbol{K}_\alpha$ and $X \subseteq A$.

$$\mathfrak{Ig}^{\mathfrak{A}} X = \{y \in A : y \leq \mathsf{c}_{(\Gamma)}(x_0 + \dots x_{k-1}) : \text{for some } x \in {}^k X \text{ and } \Gamma \subseteq_w \alpha\}.$$

The following about ideals in $\boldsymbol{K}$ algebras will be frequently used.

• If $\mathfrak{A} \subseteq \mathfrak{B}$ are algebras and I is an ideal of $\mathfrak{A}$, then $\mathfrak{Ig}^{\mathfrak{B}}(I) = \{b \in B : \exists a \in I(b \leq a)\}$.

• If I and J are ideals of an algebra, then the ideal generated by $I \cup J$ is $I + J = \{x : x \leq i + j : \text{ for some } i \in I,\ j \in J\}$.

Let

$$M = \mathfrak{Ig}^{\mathfrak{A}(X_2)}[\{s \cdot t\} \cup (J \cap A^{(X_2)})]; \tag{5}$$

$$N = \mathfrak{Ig}^{\mathfrak{A}(X_1)}[(M \cap A^{(X_1 \cap X_2)}) \cup (J \cap A^{(X_1)})]. \tag{6}$$

By (6), we have

$$N \cap A^{(X_1 \cap X_2)} = M \cap A^{(X_1 \cap X_2)}.$$

For R an ideal of $\mathfrak{A}$ and $X \subseteq A$. By $(\mathfrak{A} / R)^{(X)}$, we understand the subalgebra of $\mathfrak{A} / R$ generated by $\{x / R : x \in X\}$. Define

$$\theta : \mathfrak{A}^{(X_1 \cap X_2)} \to \mathfrak{A}^{(X_1)} / N,$$

by

$$a \mapsto a / N.$$

Then $ker\theta = N \cap A^{(X_1 \cap X_2)}$ and $Im\theta = (\mathfrak{A}^{(X_1)} / N)^{(X_1 \cap X_2)}$. It follows that

$$\overline{\theta} : \mathfrak{A}^{(X_1 \cap X_2)} / N \cap A^{(X_1 \cap X_2)} \to (\mathfrak{A}^{(X_1)} / N)^{(X_1 \cap X_2)},$$

defined by

$$a / N \cap A^{(X_1 \cap X_2)} \mapsto a / N$$

is a well defined isomorphism. Similarly

$$\overline{\psi} : \mathfrak{A}^{(X_1 \cap X_2)} / M \cap A^{(X_1 \cap X_2)} \to (\mathfrak{A}^{(X_2)} / M)^{(X_1 \cap X_2)},$$

defined by

$$a / M \cap A^{(X_1 \cap X_2)} \mapsto a / M$$

is also a well defined isomorphism. But

$$N \cap A^{(X_1 \cap X_2)} = M \cap A^{(X_1 \cap X_2)}.$$

Hence

$$\phi : (\mathfrak{A}^{(X_1)} / N)^{(X_1 \cap X_2)} \to (\mathfrak{A}^{(X_2)} / M)^{(X_1 \cap X_2)},$$

defined by

$$a / N \mapsto a / M$$

is a well defined isomorphism. Now $(\mathfrak{A}^{(X_1)} / N)^{(X_1 \cap X_2)}$ embeds into $\mathfrak{A}^{(X_1)} / N$ via the inclusion map; it also embeds in $\mathfrak{A}^{(X_2)} / M$ via $i \circ \phi$, where i is also the inclusion map. For brevity, let $\mathfrak{A}_0 = (\mathfrak{A}^{(X_1)} / N)^{(X_1 \cap X_2)}$, $\mathfrak{A}_1 = \mathfrak{A}^{(X_1)} / N$ and $\mathfrak{A}_2 = \mathfrak{A}^{(X_2)} / M$ and $j = i \circ \phi$. Then $\mathfrak{A}_0$ embeds in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ via i and j, respectively. Now observe that $\mathfrak{A}_1, \mathfrak{A}_2$ and $\mathfrak{A}_0$ are in $\boldsymbol{RK}_\alpha$. So by assumption, there exists an amalgam, i.e., there exists $\mathfrak{B} \in \boldsymbol{K}_\alpha$ and monomorphisms f and g from $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively, to $\mathfrak{B}$ such that $f \circ i = g \circ j$. Let

$$\bar{f} : \mathfrak{A}^{(X_1)} \to \mathfrak{B}$$

be defined by

$$a \mapsto f(a / N),$$

and

$$\bar{g} : \mathfrak{A}^{(X_2)} \to \mathfrak{B}$$

be defined by

$$a \mapsto g(a / M).$$

Let $\mathfrak{B}'$ be the algebra generated by $Imf \cup Img$. Then $\bar{f} \cup \bar{g} \upharpoonright X_1 \cup X_2 \to \mathfrak{B}'$ is a function, since $\bar{f}$ and $\bar{g}$ coincide on $X_1 \cap X_2$. By freeness of $\mathfrak{A}$, there exists $h : \mathfrak{A} \to \mathfrak{B}'$ such that $h\upharpoonright_{X_1 \cup X_2} = \bar{f} \cup \bar{g}$. Let $P = kerh$. Then it is not hard to check that

$$P \cap A^{(X_1)} = N, \tag{7}$$

and

$$P \cap A^{(X2)} = M. \tag{8}$$

In view of (2), (5), (8), we have $s \cdot t \in P$ and hence by (4), $r \in P$. Consequently from (2) and (7), we get $r \in N$. From (6), there exist elements

$$u \in M \cap A^{(X_1 \cap X_2)}, \tag{9}$$

and $b \in J$ such that

$$r \leq u + b. \tag{10}$$

Since $u \in M$ by (5), there is a $\Gamma \subseteq_w \alpha$ and $c \in J$ such that

$$u \leq \mathsf{c}_{(\Gamma)}(s \cdot t) + c.$$

Recall that h is the homomorphism from $\mathfrak{A}$ to $\mathfrak{Rd}_K\mathfrak{D}$ such that $h(i) = i'$ for $i \in \{x, y, w, z\}$ and $h(m) = \mathsf{d}_{01}$ and that $kerh = J$. Then $h(b) = h(c) = 0$. It follows that

$$h(r) \leq h(u) \leq \mathsf{c}_{(\Gamma)}(h(s) \cdot h(t)).$$

Let $r' = h(r)$, $u' = h(u)$, $s' = h(s)$, $t' = h(t)$. Let

$$\mathfrak{B} = (\wp({}^{\alpha}\alpha), \cup, \cap, \sim, \emptyset, {}^{\alpha}\alpha, \mathsf{C}_i, \mathsf{D}_{ij}, \mathsf{S}_{[ij]})_{i, j \in \alpha},$$

that is, $\mathfrak{B}$ is the full quasi-polyadic equality set algebra in the space ${}^{\alpha}\alpha$. Let E be the set of all equivalence relations on α, and for each $R \in E$ set

$$X_R = \{\varphi : \varphi \in {}^{\alpha}\alpha \text{ and for all } \xi, \eta < \alpha, \varphi_\xi = \varphi_\eta \text{ iff } \xi R \eta\}.$$

More succinctly

$$X_R = \{\varphi \in {}^{\alpha}\alpha : ker\varphi = R\}.$$

Let

$$C = \{\bigcup_{R \in L} X_R : L \subseteq E\}.$$

C is clearly closed under the formation of arbitrary unions, and since

$$\sim \bigcup_{R \in L} X_L = \bigcup_{R \in E \sim L} X_R$$

for every $L \subseteq E$, we see that C is closed under the formation of complements with respect to ${}^{\alpha}\alpha$. Thus C is a Boolean subuniverse (indeed, a complete Boolean subuniverse) of $\mathfrak{B}$; moreover, it is obvious that

$$X_R \text{ is an atom of } (C, \cup, \cap, \sim, 0, {}^{\alpha}\alpha) \text{ for each } R \in E. \tag{11}$$

For all $i, j \in \alpha . \mathsf{D}_{ij} = \bigcup \{X_R : (i, j) \in R\}$ and hence $\mathsf{D}_{ij} \in C$. Also,

$$\mathsf{C}_i X_R = \bigcup \{X_S : S \in E, {}^2(\alpha \sim \{i\}) \cap S ={}^2 (\alpha \sim \{i\}) \cap R\}$$

for any $i \in \alpha$ and $R \in E$. Thus, because C_i is completely additive, C is closed under the operation C_i for every $i \in \alpha$. Also it is straightforward to see that C is closed under substitutions. For any $\tau = [i, j] \in {}^{\alpha}\alpha$,

$$S_\tau X_R = \bigcup \{X_S : S \in E, \forall i, j < w(iRj \leftrightarrow \tau(i) S \tau(j)\}.$$

Therefore, we have shown that

$$C \text{ is a quasi-polyadic equality subuniverse of } \mathfrak{B}. \tag{12}$$

We now show that there is a subset Y of ${}^{\alpha}\alpha$ such that

$$X_{Id} \cap f(r') \neq 0 \text{ for every } f \in Hom(\mathfrak{D}, \mathfrak{B})$$

$$\text{such that } f(x') = X_{Id} \text{ and } f(y') = Y, \tag{13}$$

and also that for every $\Gamma \subseteq_w \alpha$, there are subsets Z, W of ${}^{\alpha}\alpha$ such that

$$X_{Id} \sim \mathsf{C}_{(\Gamma)} g(s' \cdot t') \neq 0, \text{ for every } g \in Hom(\mathfrak{D}, \mathfrak{B}),$$

$$\text{such that } g(x') = X_{Id}, \; g(z') = Z \text{ and } g(w') = W. \tag{14}$$

Here $Hom(\mathfrak{A}, \mathfrak{B})$ stands for the set of all homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$. This part is taken from Pigozzi [9] p. 340-341. Let $\sigma \in {}^{\alpha}\alpha$ be such that $\sigma_0 = 0$, and $\sigma_k = k+1$ for every non-zero $k < w$ and $\sigma\eta = \eta$ for all η, $w \leq \eta < \alpha$. Let $\tau = \sigma \restriction (\alpha \sim \{0\}) \cup \{(0, 1)\}$. Then $\sigma, \tau \in X_{Id}$. Take

$$Y = \{\sigma\}.$$

Then

$$\sigma \in X_{Id} \cap \mathsf{C}_1 Y \text{ and } \tau \in X_{Id} \sim \mathsf{C}_1 Y,$$

and hence

$$\sigma \in \mathsf{C}_0(X_{Id} \cap \mathsf{C}_1 Y) \cap \mathsf{C}_0(X_{Id} \sim \mathsf{C}_1 Y). \tag{15}$$

Therefore, we have $\sigma \in f(r)$ for every $f \in Hom(\mathfrak{A}, \mathfrak{B})$ such that $f(x) = X_{Id}$ and $f(y) = Y$, and that (13) holds. We now want to show that for any given finite $\Gamma \subseteq_w \alpha$, there exist sets $Z, W \subseteq {}^{\alpha}\alpha$ such that (14) holds; it is clear that no generality is lost if we assume that $0, 1 \in \Gamma$, so we make this assumption. Take

$$Z = \{\varphi : \varphi \in X_{Id}, \varphi_0 < \varphi_1\} \cap \mathsf{C}_{(\Gamma)}\{Id\}$$

and

$$W = \{\varphi : \varphi \in X_{Id},\ \varphi_0 > \varphi_1\} \cap \mathsf{C}_{(\Gamma)}\{Id\}.$$

It is shown in [9] p. 340, item (8) that

$$Id \in X_{Id} \sim \mathsf{C}_{(\Gamma)} g(s' \cdot t'), \tag{16}$$

for any $g \in Hom(\mathfrak{A}, \mathfrak{B})$ such that $g(x') = X_{Id}$, $g(z') = Z$, and $g(w') = W$. Note that

$$g(s' \cdot t') = g(\mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 z' \cdot \mathsf{s}_1^0 \mathsf{c}_1 z' \cdot -\mathsf{d}_{01}) + \mathsf{c}_0(x' \cdot -\mathsf{c}_1 z')$$

$$\cdot\, \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 w' \cdot \mathsf{s}_1^0 \mathsf{c}_1 w' \cdot -\mathsf{d}_{01}) + \mathsf{c}_0(x' \cdot -\mathsf{c}_1 w'))$$

$$= [\mathsf{C}_0\mathsf{C}_1(\mathsf{C}_1 Z \cap \mathsf{S}_1^0 \mathsf{C}_1 Z \sim \mathsf{D}_{01}) \cup \mathsf{C}_0(X \sim \mathsf{C}_1 Z)]$$

$$\cap\, [\mathsf{C}_0\mathsf{C}_1(\mathsf{S}_1 W \cap \mathsf{S}_1^0 \mathsf{C}_1 W \sim \mathsf{D}_{01}) \cup \mathsf{C}_0(X \sim \mathsf{C}_1 W)].$$

Now there exists a finite $\Gamma \subseteq \alpha$ and an interpolant $u' \in \mathfrak{D}^{(x')}$, that is,

$$r' \le u' \le \mathsf{c}_{(\Gamma)}(s' \cdot t').$$

There also exist $Y, Z, W \subseteq {}^{\alpha}\alpha$ such that (13) and (14) hold. Take any $k \in Hom(\mathfrak{D}, \mathfrak{B})$ such that $k(x') = X_{Id}$, $k(y') = Y$, $k(z') = Z$, and $k(w') = W$. This is possible by the freeness of $\mathfrak{D}$. Then using the fact that $X_{Id} \cap k(r')$ is non-empty by (13), we get

$$X_{Id} \cap k(u') = k(x' \cdot u') \supseteq k(x' \cdot r') \neq 0.$$

And using the fact that $X_{Id} \sim \mathsf{C}_{(\Gamma)} k(s' \cdot t')$ is non-empty by (14), we get

$$X_{Id} \sim k(u') = k(x' \cdot -u') \supseteq k(x' \cdot -\mathsf{c}_{(\Gamma)}(s' \cdot t')) \neq 0.$$

However, in view of (11), it is impossible for X_{Id} to intersect both $k(u')$ and its complement since $k(u') \in C$ and X_{Id} is an atom; to see that $k(u')$ is indeed contained in C recall that $h(u) = u' \in \mathfrak{D}^{(x')}$, and then observe that because of (12) and the fact that $X_{Id} \in C$, we must have $k[\mathfrak{D}^{(x')}] \subseteq C$. This contradiction shows that $\boldsymbol{RK}_\alpha$ does not have the amalgamation property with respect to $\boldsymbol{K}_\alpha$. By this the proof is complete.

■

Unless otherwise specified $\boldsymbol{K} \in \{\boldsymbol{SC}, \boldsymbol{QA}, \boldsymbol{CA}, \boldsymbol{QEA}\}$. For a cardinal $\beta > 0$, $L \subseteq \boldsymbol{K}_\alpha$, and $\rho : \beta \to \wp(\alpha)$, $\mathfrak{Fr}_\beta^\rho L$ stands for the dimension restricted L free algebra on β generators. The sequence $\langle \eta / Cr_\beta^\rho L : \eta < \beta \rangle$ L-freely generates $\mathfrak{Fr}_\beta^\rho L$ [6] Theorem 2.5.35. $\mathfrak{Fr}_\beta^\rho L$ is treated in [9] under the name of free algebras over L subject to certain defining relations, cf. [9] Definition 1.1.5. $\mathfrak{Fr}_\beta^\rho L$ is a quotient of the absolutely $\boldsymbol{K}$ free algebra. The following is completely analogous to Lemma 1 in [3].

Lemma 3. *If* $\alpha < \beta$ *are any ordinals and* $L \subseteq \boldsymbol{K}_\alpha$, *then, in the sequence of conditions* (1)-(5) *below*, (1)-(4) *implies the immediately following one*:

(1) *For any* $\mathfrak{A} \in L$ *and* $\mathfrak{B} \in \boldsymbol{K}_\beta$ *with* $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$, *for all* $X \subseteq A$, *we have* $\mathfrak{Sg}^{\mathfrak{A}} X = \mathfrak{Nr}_\alpha \mathfrak{Sg}^{\mathfrak{B}} X$.

(2) *For any* $\mathfrak{A} \in L$ *and* $\mathfrak{B} \in \boldsymbol{K}_\beta$ *with* $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$, *if* $\mathfrak{Sg}^{\mathfrak{B}} A = \mathfrak{B}$, *then* $\mathfrak{A} = \mathfrak{Nr}_\alpha \mathfrak{B}$.

(3) *For any* $\mathfrak{A} \in L$ *and* $\mathfrak{B} \in \boldsymbol{K}_\beta$ *with* $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$, *if* $\mathfrak{Sg}^{\mathfrak{B}} A = \mathfrak{B}$, *then for any ideal* I *of* $\mathfrak{B}$, $\mathfrak{Ig}^{\mathfrak{B}}(A \cap I) = I$.

(4) *If whenever* $\mathfrak{A} \in L$, *there exists* $x \in {}^{|A|}A$ *such that*, *if* $\rho = \langle \Delta x_i : i < |A| \rangle$, $\mathfrak{D} = \mathfrak{Fr}_{|A|}^\rho \boldsymbol{K}_\beta$ *and* $g_\xi = \xi / Cr_{|A|}^\rho \boldsymbol{K}_\beta$, *then* $\mathfrak{Sg}^{\mathfrak{Rd}_\alpha \mathfrak{D}}\{g\xi : \xi < |A|\} \in L$, *then the following hold*: *For* $\mathfrak{A}, \mathfrak{A}' \in L$, $\mathfrak{B}, \mathfrak{B}' \in \boldsymbol{K}_\beta$ *with embeddings* $e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha \mathfrak{B}$ *and* $e_{A'} : \mathfrak{A}' \to \mathfrak{Nr}_\alpha \mathfrak{B}'$ *such that* $\mathfrak{Sg}^{\mathfrak{B}} e_A(A) = \mathfrak{B}$ *and*

$\mathfrak{Sg}^{\mathfrak{B}'}e_{A'}(A) = \mathfrak{B}'$, *whenever* $i : \mathfrak{A} \to \mathfrak{A}'$ *is an isomorphism, then there exists an isomorphism* $\bar{i} : \mathfrak{B} \to \mathfrak{B}'$ *such that* $\bar{i} \circ e_A = e_{A'} \circ i$.

(5) *Assume that* $\beta = \alpha + w$. *Then L has the amalgamation property with respect to* $\boldsymbol{RK}_\alpha$.

Proof. [3], upon noting that like the $\boldsymbol{CA}$ case, the class $\boldsymbol{K} = \{A \in \boldsymbol{K}_{\alpha+w} : \mathfrak{Sg}^A \mathfrak{Nr}_\alpha \mathfrak{A} = \mathfrak{A}\}$ has AP.

Using the two previous lemmas, we infer that (1)-(5) in Lemma 3 are false for $\boldsymbol{RK}_\alpha$ when $\alpha \geq w$, thus we arrive at the results in the abstract for ***SC****'s* ***QA****'s* and ***QEA****'s*. In more detail, we have:

Theorem 4. *For* $\alpha \geq w$ *and* $\boldsymbol{K} \in \{\boldsymbol{SC}, \boldsymbol{CA}, \boldsymbol{QA}, \boldsymbol{QEA}\}$, *the following hold*:

(i) *There are non-isomorphic* $\boldsymbol{RK}_\alpha$*'s each of which is a generating subreduct of the same* $\alpha + w\ \boldsymbol{K}$ *algebra.*

(ii) *There exist* $\mathfrak{A} \in \boldsymbol{K}_\alpha$, *a* $\mathfrak{B} \in \boldsymbol{K}_{\alpha+w}$ *and an ideal* $J \subseteq \mathfrak{B}$, *such that* $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$, *A generates* $\mathfrak{B}$, *but* $\mathfrak{Ig}^{\mathfrak{B}}(J \cap A) \neq \mathfrak{B}$.

(iii) *There exist* $\mathfrak{A}, \mathfrak{A}' \in \boldsymbol{K}_\alpha$, $\mathfrak{B}, \mathfrak{B}' \in \boldsymbol{K}_{\alpha+\beta}$ *with embeddings* $e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha \mathfrak{B}$ *and* $e_{A'} : \mathfrak{A}' \to \mathfrak{Nr}_\alpha \mathfrak{B}'$ *such that* $\mathfrak{Sg}^{\mathfrak{B}} e_A(A) = \mathfrak{B}$ *and* $\mathfrak{Sg}^{\mathfrak{B}'}e_{A'}(A) = \mathfrak{B}'$, *and an isomorphism* $i : \mathfrak{A} \to \mathfrak{A}'$ *for which there exists no isomorphism* $\bar{i} : \mathfrak{B} \to \mathfrak{B}'$ *such that* $\bar{i} \circ e_A = e_{A'} \circ i$.

In contrast, we show that ***PA****'s* behave differently; in fact, they behave like ***Dc****'s*. For undefined terminology in the coming theorem, the reader is referred to [5] and [8]. If $\alpha < \beta$, $\mathfrak{B} \in \boldsymbol{PA}_\beta$ is a minimal dilation of $\mathfrak{A} \in \boldsymbol{PA}_\alpha$, if $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$ and A generates $\mathfrak{B}$.

Theorem 5. *Suppose that J is an infinite set*, $\alpha < \beta$ *are ordinals*, $J \subseteq \alpha \subseteq \beta$ *and* $\mathfrak{A} \in \boldsymbol{PA}_\alpha$.

(i) *If* $\mathfrak{B}$ *is a minimal* β *dilation of* $\mathfrak{A}$, *then for all* $X \subseteq A$, $\mathfrak{Sg}^{\mathfrak{A}}X = \mathfrak{Nr}_{\alpha}\mathfrak{Sg}^{\mathfrak{B}}X$ *and* $\mathfrak{A}$ *is a faithful compression of* $\mathfrak{B}$.

(ii) $\mathfrak{Nr}_J\mathfrak{B}$ *is a faithful compression of* $\mathfrak{A}$ *iff* $|J| \geq \mathfrak{n}$, *where* $\mathfrak{n}$ *is the cardinal predecessor of the local degree* $\mathfrak{m}$ *of* $\mathfrak{A}$. *That is* $\mathfrak{n} = \mathfrak{m}$, *if* $\mathfrak{m}$ *is a limit cardinal and* $\mathfrak{n}^+ = \mathfrak{m}$ *if* $\mathfrak{m}$ *is a successor cardinal.*

Proof. (ii) is stated in [8] and proved in [5]. Here we prove (i) which is not proved in the cited papers. Abusing notation, we write $\mathfrak{A}$ for $\mathfrak{Sg}^{\mathfrak{A}}X$ and $\mathfrak{B}$ for $\mathfrak{Sg}^{\mathfrak{B}}X$. Then $\mathfrak{B}$ is a minimal dilation of $\mathfrak{A}$. By [5], Theorem 3.3, each element of $\mathfrak{B}$ has the form $\mathsf{s}_{\sigma}^{\mathfrak{B}}a$ for some $a \in A$, and σ a transformation on β such that $\sigma \restriction \alpha$ is one to one. We claim that $\mathfrak{Nr}_{\alpha}\mathfrak{B} \subseteq \mathfrak{A}$. Indeed, let $x \in Nr_{\alpha}B$. Then by the above, we have $x = \mathsf{s}_{\sigma}^{\mathfrak{B}}y$, for some $y \in A$ and $\sigma \in {}^{\beta}\beta$. Let $\tau \in {}^{\beta}\beta$ such that

$$\tau \restriction \alpha \subseteq Id \text{ and } (\tau\, o\, \sigma)\alpha \subseteq \alpha. \tag{17}$$

Such a τ clearly exists. Since $x \in \mathfrak{Nr}_{\alpha}\mathfrak{B}$, it follows by definition that $\mathsf{c}_{(\beta \sim \alpha)}x = x$. From

$$\tau \restriction \beta \sim (\beta \sim \alpha) = \tau \restriction \alpha = Id \restriction \alpha = Id \restriction \beta \sim (\beta \sim \alpha),$$

we get from the polyadic axioms that

$$\mathsf{s}_{\tau}^{\mathfrak{B}}x = \mathsf{s}_{\tau}^{\mathfrak{B}}\mathsf{c}_{(\beta\sim\alpha)}x = \mathsf{s}_{Id}^{\mathfrak{B}}\mathsf{c}_{(\beta\sim\alpha)}x = \mathsf{s}_{Id}^{\mathfrak{B}}x = x.$$

Therefore

$$x = s_{\tau}^{\mathfrak{B}}x = s_{\tau}^{\mathfrak{B}}s_{\sigma}^{\mathfrak{B}}x = s_{\tau o \sigma}^{\mathfrak{B}}x. \tag{18}$$

Let

$$\mu = \tau\, o\, \sigma \restriction \alpha \text{ and } \overline{\mu} = \mu \cup Id \restriction (\beta \sim \alpha).$$

Since

$$\overline{\mu} \restriction \beta \sim (\beta \sim \alpha) = \overline{\mu} \restriction \alpha = \mu = \tau \text{ o } \sigma \restriction \beta \sim (\beta \sim \alpha),$$

we have

$$\mathsf{s}_{\overline{\mu}}^{\mathfrak{B}}\mathsf{c}_{(\beta\sim\alpha)}y = \mathsf{s}_{\tau o \sigma}^{\mathfrak{B}}\mathsf{c}_{(\beta\sim\alpha)}y.$$

Since $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$ and $y \in A$, we have $\mathsf{s}_\mu^{\mathfrak{A}} y = \mathsf{s}_{\overline{\mu}}^{\mathfrak{B}} y$ and $\mathsf{c}_{(\beta \sim \alpha)}^{\mathfrak{B}} y = y$. Therefore

$$\mathsf{s}_\mu^{\mathfrak{A}} y = \mathsf{s}_{\overline{\mu}}^{\mathfrak{B}} y = \mathsf{s}_{\overline{\mu}}^{\mathfrak{B}} \mathsf{c}_{(\beta \sim \alpha)}^{\mathfrak{B}} y = \mathsf{s}_{\tau o \sigma}^{\mathfrak{B}} \mathsf{c}_{(\beta \sim \alpha)}^{\mathfrak{B}} y = \mathsf{s}_{\tau o \sigma}^{\mathfrak{B}} y. \tag{19}$$

From (18) and (19), we get $x = \mathsf{s}_\mu^{\mathfrak{A}} y \in \mathfrak{A}$. By this the proof is complete, since x was arbitrary.

■

To summarize, we have Theorems 2.6.67 (ii), 2.6.71-72 of [6] formulated for ***Dc****'s* do not hold for $\boldsymbol{K} \in \{\boldsymbol{SC}, \boldsymbol{CA}, \boldsymbol{QA}, \boldsymbol{QEA}\}$, in fact they do not hold for $\boldsymbol{RK}_\alpha$ but they hold for $\boldsymbol{PA}_\alpha$'s. Here α is an infinite ordinal. This establishes yet another dichotomy between the ***CA*** paradigm and the ***PA*** paradigm.

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